Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 10.06.2013.

Pismeni ispit iz predmeta Linearna algebra

Bitna napomena: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. Neka je $\mathcal{M} = \{X \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid AX = \mathbf{0}\}$ gdje je $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Odrediti bazu i dimenziju vektorskog prostora $\mathcal{M}$. Odredite mu i neku bazu za (direktni) komplement (koji nije otrogonalni komplement).

2. (20%)(a) Dat je linearni operator

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x \longrightarrow Ax$$

gdje je $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. Odredite jezgru i sliku od T i izračunati njihove dimenzije.

(80%)(b) Zadan je linearni operator $T : \mathbb{R}^3 \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{R})$ sa

$$T(a, b, c) = \begin{pmatrix} a + 2b & a - b + c \\ b - 2c & 5a + 5c \end{pmatrix}.$$

Prikažite operator T (odredite matricu koordinata) u paru standardnih baza, te mu odredite po jednu bazu za jezgru i sliku.

3. Neka je T linearni operator na prostoru $\mathbb{R}^2$ koji vektoru pridružuje njegovu ortogonalnu projekciju na pravac $y = -x$. Odredite djelovanje linearnog operatora T na proizvoljnom vektoru $\begin{pmatrix} x \\ y \end{pmatrix}$ i odredite mu matrični prikaz u standardnoj bazi, te u bazi $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$. Odrediti i matricu koordinata operatora T u odnosu na standardnu bazu, te u odnosu na bazu $\mathcal{B}$.

4. U unitarnom prostoru $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ sa standardnim skalarnim (unutrašnjim) proizvodom $\langle A, B\rangle = \mathrm{trag}(A^\top B)$ dat je podprostor

$$\mathcal{M} = \mathrm{span}\left\{ \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Odredite jednu bazu za $\mathcal{M}^\perp$, te nađite prikaz matrice $A = \begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix}$ u obliku $A = B + C$, gdje je $B \in \mathcal{M}$, $C \in \mathcal{M}^\perp$.

Zadaci su skinuti sa stranice pf.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Neka je $M = \{ X \in Mat_{2\times2}(\mathbb{R}) \mid AX = 0 \}$ gdje je $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Odrediti bazu i dimenziju vektorskog prostora M. Odredite mu i neku bazu za (direktni) kompliment (koji nije ortogonalni komplement).

Rj.

$$M = \{ X \in Mat_{2\times2}(\mathbb{R}) \mid AX = 0 \} =$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat_{2\times2} \;\middle|\; \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_{2\times2}(\mathbb{R}) \;\middle|\; 2a+c=0,\ 2b+d=0,\ 6a+3c=0,\ 6b+3d=0 \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_{2\times2}(\mathbb{R}) \;\middle|\; \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$ pa da bi odredili nepoznate a, b, c, d dovoljno je odrediti jezgru od matrice B:

$$\ker \underbrace{\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 \end{pmatrix}}_{=B} = \ker(B)$$

Generator skup za $\ker(B)$ su u stvari vektori iz opšteg rješenja jednačine $Bx = 0$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 \end{pmatrix} \underset{IV_v + II_v \cdot (-3)}{\overset{III_v + I_v \cdot (-3)}{\sim}} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left. \begin{matrix} rang(B) = 2 \\ rang(\overline{B}) = 2 \end{matrix} \right\} \Rightarrow$$

$\Rightarrow$ sistem $Bx = 0$ ima ∞ mnogo rješenja i dvije promjenjive uzimamo proizvoljno

$2x_1 + x_3 = 0 \qquad x_3 = s \qquad x_1 = -\frac{s}{2}$

$2x_2 + x_4 = 0 \qquad x_4 = t \qquad x_2 = -\frac{t}{2}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} t$$

$$\Rightarrow \quad M = span\left\{\begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}\right\}$$

Baza za vektorski prostor M je $\left\{\begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}\right\}$ iz čega slijedi da je $\dim M = 2$.

Prisjetimo se

Komplementarni podprostor

Za podprostore X, Y prostora V kažemo da su komplementarni akko je $V = X + Y$ i $X \cap Y = 0$ i u tom slučaju kažemo da je V direktna suma od X i Y, i ovo označavamo sa $V = X \oplus Y$.

$X + Y := \{x + y \mid x \in X \text{ i } y \in Y\}$.

Ako X i Y imaju redom baze $\mathcal{B}_X$ i $\mathcal{B}_Y$ vrijedi sljedeće

$V = X \oplus Y \Leftrightarrow \forall v \in V \ \exists!\, x \in X, y \in Y$ $v = x + y$ $\Leftrightarrow$ $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$, $\mathcal{B}_X \cup \mathcal{B}_Y$ je baza za V

Da bi smo odredili komplement od M prvo ćemo odrediti komplement od $\ker(B)$ tj. nadopunićemo skup $\left\{\begin{pmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}\right\}$ do baze za $\mathbb{R}^4$.

$$\begin{pmatrix} -1/2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{II_v \cdot (-2)}{\overset{I_v \cdot (-2)}{\sim}} \begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{IV_v + II_v \cdot (-1)}{\overset{III_v + I_v \cdot (-1)}{\sim}} \begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

$$\overset{III_v:2}{\underset{IV_v:2}{\sim}} \begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 1 \end{pmatrix}$$

Komplement od $\ker(B)$ je $\text{span}\left\{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Prema tome (direktni) komplement od M je

$$N = \text{span}\left\{\begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}\right\}$$

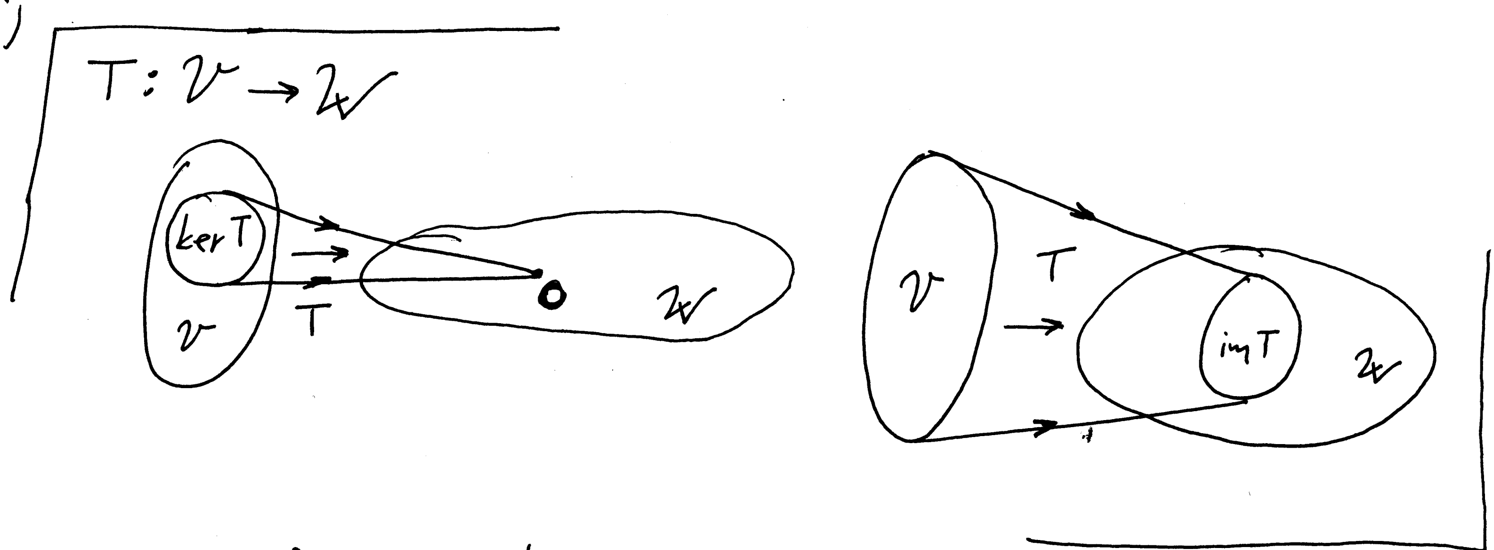
↑ baza za direktni komplement

Dat je linearni operator

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\mathbf{x} \to A\mathbf{x}$$

gdje je $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Odrediti jezgru i sliku od T i izračunati njihove dimenzije.

Rj:



$$\ker(T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; T(\mathbf{x}) = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; A\mathbf{x} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \ker(A)$$

Jezgro od matrice A čine vektori iz opšteg rješenja sistema $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \overset{II_v + I_v \cdot (-2)}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow \ker(A) = \ker(\bar{A}) = 1 < 2$$

jednu promjenjivu uzimamo proizvoljno

$x + 3y = 0$

$x = -3y$

$y = t \Rightarrow x = -3t$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3t \\ t \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} t \Rightarrow \ker(T) = \operatorname{span}\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$$

Dimenzija od $\ker(T)$ je 1.

$$\operatorname{im}(T) = \{Tx \mid x \in \mathbb{R}^2\} = \{Ax \mid x \in \mathbb{R}^2\} = \operatorname{im}(A)$$

Kako je $A \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ to je

$$\operatorname{im}(T) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$$

i dimenzija od slike od T je 1.

(#) Zadan je linearni operator $T : \mathbb{R}^3 \longrightarrow Mat_{2\times 2}(\mathbb{R})$ sa

$$T(a,b,c) = \begin{pmatrix} a+2b & a-b+c \\ b-2c & 5a+5c \end{pmatrix}.$$

Prikažite operator T (odredite matricu koordinata) u paru standardnih baza, temu odredite po jednu bazu za jezgru i sliku.

R: Standardnu bazu za $\mathbb{R}^3$ označimo sa $\varphi = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

a standardnu bazu za $Mat_{2\times 2}(\mathbb{R})$ označimo sa $\varphi' = \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$.

R: Matrica koordinata

Neka su $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ i $\mathcal{B}' = \{v_1, v_2, \dots, v_n\}$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Matrica koordinata od $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ u odnosu na par $(\mathcal{B}, \mathcal{B}')$ je definisana kao $m \times n$ matrica

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

Pa da bi odredili $[T]_{\varphi\varphi'}$ trebaju nam $\left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_{\varphi'}$, $\left[T\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_{\varphi'}$ i $\left[T\begin{pmatrix}0\\0\\1\end{pmatrix}\right]_{\varphi'}$.

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1&1\\0&5\end{pmatrix} \Rightarrow \left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_{\varphi'} = \begin{pmatrix}1\\1\\0\\5\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2&-1\\1&0\end{pmatrix} \Rightarrow \left[T\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_{\varphi'} = \begin{pmatrix}2\\-1\\1\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0&1\\-2&5\end{pmatrix} \Rightarrow \left[T\begin{pmatrix}0\\0\\1\end{pmatrix}\right]_{\varphi'} = \begin{pmatrix}0\\1\\-2\\5\end{pmatrix}$$

Tražena matrica koordinata operatora T je

$$[T]_{\varphi\varphi'} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \\ 5 & 0 & 5 \end{pmatrix}$$

Dalje, prisjetimo se

<u>Neka je $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ i neka su $\mathcal{B}$ i $\mathcal{B}'$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Tada za $\forall u \in \mathcal{U}$ imamo $[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} [u]_{\mathcal{B}}$.</u>

$$\ker(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; T(a,b,c) = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} a+2b & a-b+c \\ b-2c & 5a+5c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; a+2b=0,\ a-b+c=0,\ b-2c=0,\ 5a+5c=0 \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \\ 5 & 0 & 5 \end{pmatrix}}_{=A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \right\} = \ker(A)$$

$$= \ker([T]_{\varphi\varphi'})$$

Generator skup za $\ker(A)$ su vektori iz općeg rješenja jednačine $Ax = 0$

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \\ 5 & 0 & 5 \end{pmatrix} \underset{IV_v - I_v \cdot (5)}{\overset{II_v - I_v}{\sim}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & -10 & 5 \end{pmatrix} \overset{II_v \leftrightarrow III_v}{\sim} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & -3 & 1 \\ 0 & -10 & 5 \end{pmatrix} \underset{IV_v + II_v \cdot 10}{\overset{III_v + II_v \cdot 3}{\sim}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \\ 0 & 0 & -15 \end{pmatrix}$$

$$\underset{III_v : (-5)}{\overset{IV_v - III_v \cdot 3}{\sim}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \ker(A) = 3 \\ \ker(\bar{A}) = 3 \end{matrix} \Rightarrow \text{sistem } Ax=0 \text{ ima jedinstveno rješenje}$$

$$\begin{aligned} x_1+2x_2 \quad &= 0 \\ x_2-2x_3 &= 0 \\ x_3 &= 0 \end{aligned} \quad \Rightarrow \quad x_1=x_2=x_3=0$$

$\Rightarrow \quad \ker(A)=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ tj. $\ker(T)=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

Baza za jezgro od T je prazan skup.

$$im(T)=\left\{T\begin{pmatrix}a\\b\\c\end{pmatrix} \;\middle|\; \begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3\right\}=\left\{\begin{pmatrix}a+2b & a-b+c\\ b-2c & 5a+5c\end{pmatrix}\in Mat_{2\times2}(\mathbb{R}) \;\middle|\; \begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3\right\}$$

$$W=\left\{\begin{pmatrix}1&2&0\\1&-1&1\\0&1&-2\\5&0&5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \;\middle|\; \begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3\right\}=im(A)$$

gdje je $A=\begin{pmatrix}1&2&0\\1&-1&1\\0&1&-2\\5&0&5\end{pmatrix}$

Maloprije smo pokazali da je $A\sim\begin{pmatrix}1&2&0\\0&1&-2\\0&0&1\\0&0&0\end{pmatrix}$

$\Rightarrow \quad rang(A)=3 \quad \Rightarrow \quad im(A)=span\left\{\begin{pmatrix}1\\1\\0\\5\end{pmatrix},\begin{pmatrix}2\\-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\-2\\5\end{pmatrix}\right\}$

$\Rightarrow \quad im(T)=span\left\{\begin{pmatrix}1&1\\0&5\end{pmatrix},\begin{pmatrix}2&-1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\-2&5\end{pmatrix}\right\}$

Baza za $im(T)$ je $\left\{\begin{pmatrix}1&1\\0&5\end{pmatrix},\begin{pmatrix}2&-1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\-2&5\end{pmatrix}\right\}$.

(#) Neka je T linearni operator na prostoru $\mathbb{R}^2$ koji vektoru pridružuje njegovu ortogonalnu projekciju na pravac $y=-x$. Odredite djelovanje linearnog operatora T na proizvoljnom vektoru $\begin{pmatrix} x \\ y \end{pmatrix}$ i odredite mu matrični prikaz u standardnoj bazi, te u bazi $\mathcal{B}=\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.
Odrediti i matricu koordinata operatora T u odnosu na standardnu bazu, te u odnosu na bazu $\mathcal{B}$.

R: Prisjetimo se

Matrica koordinata

Neka su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_n\}$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Matrica koordinata od $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$ u odnosu na par $(\mathcal{B},\mathcal{B}')$ je definisana kao $m\times n$ matrica

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

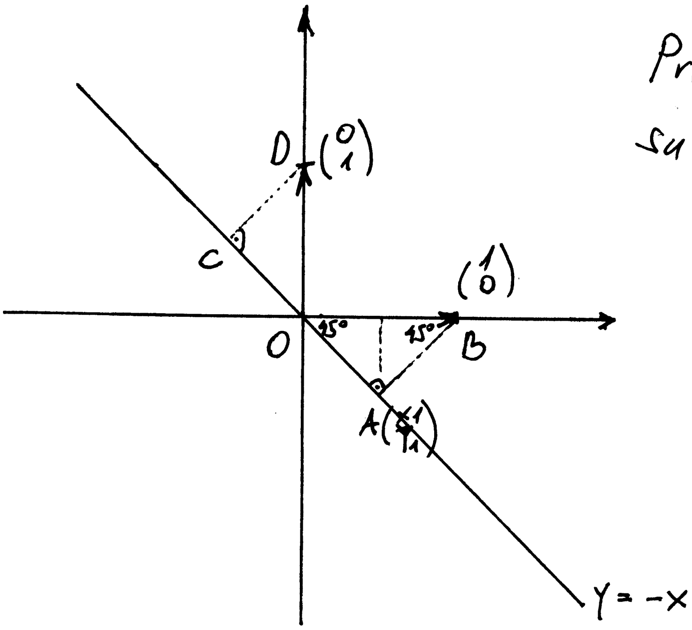
Kada je T linearni operator na $\mathcal{U}$, tada je u igri samo jedna baza, i koristimo $[T]_{\mathcal{B}}$ umjesto $[T]_{\mathcal{B}\mathcal{B}}$.

Neka je $\mathcal{S}$ standardna baza $\mathcal{S}=\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ a sa $\mathcal{B}$ označimo bazu $\mathcal{B}=\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.
U zadatku se traži da odredimo

$$[T]_{\mathcal{S}} = \begin{pmatrix} | & | \\ [T\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_{\mathcal{S}} & [T\begin{pmatrix} 0 \\ 1 \end{pmatrix}]_{\mathcal{S}} \\ | & | \end{pmatrix} \quad i \quad [T]_{\mathcal{B}} = \begin{pmatrix} | & | \\ [T\begin{pmatrix} 1 \\ 1 \end{pmatrix}]_{\mathcal{B}} & [T\begin{pmatrix} 2 \\ 1 \end{pmatrix}]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Odredimo pro $[T]_\varphi$.

Neka je $A\binom{x_1}{y_1}$ ortogonalna projekcija od $\vec{OB}=\binom{1}{0}$ na pravu



Primjetimo da je ΔOAB pravougli sa uglom $\sphericalangle AOB = 45°$ pa je i $\sphericalangle ABO = 45°$.

$A\binom{x_1}{y_1}$

$|\vec{OB}| = 1$

A leži na simetrali duži OB pa je $x_1 = \frac{1}{2}$

$O(0;0)$

$A_1(\frac{1}{2}, y_1)$

$\vec{OA_1} = (\frac{1}{2}, y_1)$

$|\vec{OA_1}| = \sqrt{\frac{1}{4}+y_1^2}$

$$|\vec{OB}|^2 = |\vec{OA}|^2 + \underbrace{|\vec{AB}|^2}_{=\vec{OA}}$$

$$2\left(\frac{1}{4}+y_1^2\right) = 1$$

$$\frac{1}{2}+2y_1^2 = 1$$

$$2y_1^2 = \frac{1}{2} \quad /\cdot\frac{1}{2}$$

$$y_1^2 = \frac{1}{4} \Rightarrow y_1 = \pm\frac{1}{2}$$

Prema tome $A\binom{1/2}{-1/2}$

Na sličan način izračunamo da je $C\binom{-1/2}{1/2}$

Prema tome imamo

$$T\binom{1}{0} = \binom{1/2}{-1/2} = \frac{1}{2}\binom{1}{0} - \frac{1}{2}\binom{0}{1} \Rightarrow \left[T\binom{1}{0}\right]_\varphi = \binom{1/2}{-1/2}$$

$$T\binom{0}{1} = \binom{-1/2}{1/2} = -\frac{1}{2}\binom{1}{0} + \frac{1}{2}\binom{0}{1} \Rightarrow \left[T\binom{0}{1}\right]_\varphi = \binom{-1/2}{1/2}$$

$$[T]_\varphi = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

matrica koordinata u odnosu na standardnu bazu

Prisjetimo se

Neka je $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ i neka su $\mathcal{B}$ i $\mathcal{B}'$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Tada za $\forall u \in \mathcal{U}$ imamo $[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} [u]_{\mathcal{B}}$.

Prema tome $\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}}$ možemo izračunati pomoću sljedeće formule
$$\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}} = [T]_{\varphi\mathcal{B}} \cdot \left[\begin{pmatrix}1\\1\end{pmatrix}\right]_{\varphi}$$

gdje je
$$[T]_{\varphi\mathcal{B}} = \begin{pmatrix} | & | \\ \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}} & \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Odredimo prvo $\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}}$ i $\left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}}$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\end{pmatrix} \qquad x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\end{pmatrix} \Rightarrow \begin{matrix} x = -\frac{3}{2} \\ y = 1 \end{matrix}$$

$$\Rightarrow \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}-3/2\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\end{pmatrix} \qquad x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\end{pmatrix} \Rightarrow \begin{array}{r} x + 2y = -\frac{1}{2} \\ - \quad x + y = 1/2 \\ \hline y = -1 \\ x = \frac{3}{2} \end{array}$$

$$\Rightarrow \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}3/2\\-1\end{pmatrix}$$

Prema tome

$$[T]_{\varphi\mathcal{B}} = \begin{pmatrix} -3/2 & 3/2 \\ 1 & -1 \end{pmatrix}$$

$$\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}-3/2 & 3/2\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}-3/2 & 3/2\\ 1 & -1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-3/2\\1\end{pmatrix}$$

Kako je $[T]_{\mathcal{B}} = \left(\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}} \; \left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_{\mathcal{B}} \right)$ to je

$$[T]_{\mathcal{B}} = \begin{pmatrix}0 & -3/2\\ 0 & 1\end{pmatrix}$$ tražena matrica koordinata u odnosu na bazu $\mathcal{B}$

Odredimo još djelovanje operatora T na proizvoljnom vektoru $\begin{pmatrix}x\\y\end{pmatrix}$ u odnosu na dvije date baze

$$\left[T\begin{pmatrix}x\\y\end{pmatrix}\right]_{\varphi} = [T]_{\varphi}\left[\begin{pmatrix}x\\y\end{pmatrix}\right]_{\varphi} = \begin{pmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{x-y}{2}\\ \frac{y-x}{2}\end{pmatrix}$$

$$\left[T\begin{pmatrix}x\\y\end{pmatrix}\right]_{\mathcal{B}} = [T] \left[\begin{pmatrix}x\\y\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}0 & -3/2\\ 0 & 1\end{pmatrix}\begin{pmatrix}2y-x\\ x-y\end{pmatrix} =$$

$$= \begin{pmatrix}-\frac{3}{2}(x-y)\\ x-y\end{pmatrix}$$

$$\alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$$\alpha + 2\beta = x$$
$$\alpha + \beta = y$$

$$\beta = x - y$$
$$\alpha + (x-y) = y$$
$$\alpha = 2y - x$$

(#) U unitarnom prostoru $Mat_{2\times2}(\mathbb{R})$ sa standardnim skalarnim (unutrašnjim) proizvodom $\langle A,B\rangle = trag(A^TB)$ dat je podprostor

$$\mathcal{M} = span\left\{\begin{pmatrix}2 & 1\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$$

Odrediti jednu bazu za $\mathcal{M}^\perp$, te nađite prikaz matrice $A=\begin{pmatrix}4 & -2\\ -1 & 2\end{pmatrix}$ u obliku $A=B+C$, gdje je $B\in\mathcal{M}$, $C\in\mathcal{M}^\perp$.

Rj. Prisjetimo se

Ortogonalni komplement

Za podskup $\mathcal{M}$ unitarnog prostora V, ortogonalni komplement $\mathcal{M}^\perp$ od $\mathcal{M}$ je definisan sa

$$\mathcal{M}^\perp = \{x\in V \mid \langle m,x\rangle = 0 \text{ za } \forall m\in\mathcal{M}\}$$

Mi u stvari tražimo koeficijente a,b,c,d tako da je

$$\left\langle\begin{pmatrix}2 & 1\\ 0 & -1\end{pmatrix}, \begin{pmatrix}a & b\\ c & d\end{pmatrix}\right\rangle = 0 \quad i \quad \left\langle\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}a & b\\ c & d\end{pmatrix}\right\rangle = 0.$$

$$trag\left(\begin{bmatrix}2 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = 0$$

$$trag\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = 0$$

$$2a+b-d=0$$
$$c+b=0$$

dvije promjenjive uzimamo proizvoljno

$b=t,\ a=s \implies$

$\implies c=-t \quad d=2s+t$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}s & t\\ -t & 2s+t\end{pmatrix} =$$

$$= \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}s + \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}t$$

$s,t\in\mathbb{R}$

Prema tome imamo $\mathcal{M}^{\perp} = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}$.

Sad želimo pronaći α, β, γ i δ tako da

$$\begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix} = \alpha \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \delta \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Ova jednakost se svodi na sistem $A \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -1 \\ 2 \end{pmatrix}$ gdje je

$$\overline{A} = \left(\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 4 \\ 1 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 & 2 \end{array} \right) \underset{I_v \leftrightarrow II_v}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -2 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 & 2 \end{array} \right) \underset{\substack{II_v + I_v \cdot (-2) \\ IV_v + I_v}}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -2 \\ 0 & -2 & 1 & -2 & 8 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 & 0 \end{array} \right)$$

$$\underset{II_v \leftrightarrow III_v}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & -2 & 1 & -2 & 8 \\ 0 & 1 & 2 & 2 & 0 \end{array} \right) \underset{\substack{III_v + II_v \cdot 2 \\ IV_v + II_v \cdot (-1)}}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -4 & 6 \\ 0 & 0 & 2 & 3 & 1 \end{array} \right) \underset{IV_v + III_v \cdot (-2)}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 11 & -11 \end{array} \right)$$

$$\underset{IV_v : 11}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right) \underset{\substack{III_v + IV_v \cdot 4 \\ II_v + IV_v \\ I_v + IV_v \cdot (-1)}}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right) \underset{I_v - II_v}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right)$$

Prema tome $\alpha = 1$, $\beta = -2$, $\gamma = 2$, $\delta = -1$ i imamo

$$\begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} - 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$